

SOLUTIONS OF EXERCISE SHEET 7

Exercise 1. Recall that holomorphicity of the functions f and g implies that

$$f(z+h) - f(z) = f'(z)h + \varepsilon_f(h)$$

and

$$g(z+h) - g(z) = g'(z)h + \varepsilon_g(h)$$

where

$$\lim_{h \rightarrow 0} \frac{\varepsilon_f(h)}{h} = \lim_{h \rightarrow 0} \frac{\varepsilon_g(h)}{h} = 0.$$

We set $h_g(h) = g'(z)h + \varepsilon_g(h)$. Then

$$\begin{aligned} f(g(z+h)) &= f(g(z) + g'(z)h + \varepsilon_g(h)) \\ &= f(g(z) + h_g(h)) \\ &= f(g(z)) + f'(g(z))h_g(h) + \varepsilon_f(h_g(h)) \\ &= f(g(z)) + f'(g(z))g'(z)h + f'(g(z))\varepsilon_g(h) + \varepsilon_f(h_g(h)). \end{aligned}$$

Consequently, as

$$\lim_{h \rightarrow 0} \frac{f'(g(z))\varepsilon_g(h)}{h} = 0,$$

we only need to show that $\varepsilon_f(h_g) \rightarrow 0$ as $h \rightarrow 0$. To that end, we note that if $h_g = 0$ then the claim is immediate. Otherwise, we have that

$$\lim_{h \rightarrow 0} h_g(h) = 0$$

which implies

$$\lim_{h \rightarrow 0} \frac{\varepsilon_f(h_g(h))}{h_g(h)} = 0.$$

Putting things together, we see that

$$f(g(z+h)) - f(g(z)) = f'(g(z))g'(z)h + o(h)$$

which yields the claim.

Exercise 2. Recall from Monday that a C^1 function f is holomorphic at a point $z = x + iy \in \mathbb{C}$ if and only if $\bar{\partial}f(z) = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f(z) = 0$. Moreover, one readily computes that

$$\bar{\partial}\bar{z}^n = n\bar{z}^{n-1}$$

So, as $\bar{\partial}(z-1) = 0$, the product rule¹ implies

$$\bar{\partial}\bar{z}^n(z-1) = n\bar{z}^{n-1}(z-1).$$

¹Strictly speaking, we have not proven a product rule for $\bar{\partial}$. However, such a product rule follows in the same fashion as the one we established on Monday

Consequently,

- $(z - 1)$ is holomorphic for all $z \in \mathbb{C}$.
- $\bar{z}(z - 1)$ is holomorphic if and only if $z = 1$
- $n \geq 2 \implies \bar{z}^n(z - 1)$ is holomorphic if and only if $z = 0, 1$.

Exercise 3. Let $f_i(t) = f(\gamma_i(t))$. Then, by definition,

$$f'_i(0) = \lim_{h \rightarrow 0} \frac{f(\gamma(h)) - f(\gamma(0))}{h}.$$

Thus, as $\gamma'_i(0) \neq 0$ and since f is holomorphic, we have that

$$\begin{aligned} f'_i(0) &= \lim_{h \rightarrow 0} \frac{f(\gamma(h)) - f(\gamma(0))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(\gamma(h)) - f(\gamma(0))}{\gamma(h) - \gamma(0)} \frac{\gamma(h) - \gamma(0)}{h} = f'(\gamma(0))\gamma'_i(0) \end{aligned}$$

and the claim follows by plugging this expression into the formula for the angle.

Exercise 4. We readily compute that

$$\int_{\gamma_1} z^2 dz = \int_0^1 (t + i)^2 dt = \frac{(1 + i)^3}{3} - \frac{i^3}{3} = -\frac{2}{3} + i$$

and

$$\begin{aligned} \int_{\gamma_1} z^2 dz &= \int_0^1 (\cos(\pi t) + i \sin(\pi t))^2 (-\sin(\pi t) + i \cos(\pi t)) \pi dt \\ &= i\pi \int_0^1 (\cos(\pi t) + i \sin(\pi t))^2 (\sin(\pi t) + \cos(\pi t)) dt \\ &= i\pi \int_0^1 e^{3\pi i t} dt = -\frac{2}{3}. \end{aligned}$$